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# The Geometry of Developing Flame Fronts: Analysis with Pole Decomposition

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The roughening of expanding flame fronts by the accretion of cusp-like singularities is a fascinating example of the interplay between instability, noise and nonlinear dynamics that is reminiscent of self-fractalization in Laplacian growth patterns. The nonlinear integro-differential equation that describes the dynamics of expanding flame fronts is amenable to analytic investigations using pole decomposition. This powerful technique allows the development of a satisfactory understanding of the qualitative and some quantitative aspects of the complex geometry that develops in expanding flame fronts.

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The study of growing fronts in nonlinear physics [1] offers fascinating examples of spontaneous generation of fractal geometry [2,3]. Advancing fronts rarely remain flat; usually they form either fractal objects with contorted and ramified appearance, like Laplacian growth patterns and diffusion limited agregates (DLA) [4], or they remain graphs, but they "roughen" in the sense of producing self-affine fractals whose "width" diverges with the linear scale of the system with some characteristic exponent. The study of interface growth where the roughening is caused by the noisy environment, with either annealed or quenched noise, was a subject of active research in recent years [5,6]. These studies met considerable success and there is significant analytic understanding of the nature of the universality classes that can be expected. The study of interface roughening in system in which the flat surface is inherently unstable is less developed. One interesting example that attracted attention is the Kuramoto-Sivashinsky equation [7,8] which is known to roughen in 1+1 dimensions but is claimed not to roughen in higher dimensions [9]. Another outstanding example is Laplacian growth patterns [10]. This Letter is motivated by a new example of the dynamics of outward propagating flames whose front wrinkles and fractalizes [11]. We will see that this problem has many features that closely resemble Laplacian growth, including the existence of a single finger in channel growth versus tip splitting in cylindrical outward growth, extreme sensitivity to noise, etc. In the case of flame fronts the equation of motion is amenable to analytic solutions and as a result we can understand some of these issues.

The physical problem that motivates this analysis is that of pre-mixed flames which exist as self-sustaining fronts of exothermic chemical reactions in gaseous combustion. It had been known for some time that such flames are intrinsically unstable [12]. It was reported that such flames develop characteristic structures which includes cusps, and that under usual experimental conditions the flame front accelerates as time goes on [13]. In recent work Filyand et al. [11] proposed an equation of motion that is motivated by the physics and seems to capture a number of the essential features of the observations. The equation is written in cylindrical geometry and is for $R(\theta,t)$ which is the modulus of the radius vector on the flame front:

$$\frac{\partial R}{\partial t}=\frac{U_b}{2{R_0}^2(t)}\left(\frac{\partial R}{\partial\theta}\right)^2+\frac{D_M}{{R_0}^2(t)}\frac{\partial^2 R}{\partial\theta^2} + \frac{\gamma U_b}{2R_0(t)}I(R)+U_b\ . \tag{1}$$

Here $0<\theta<2\pi$ is an angle and the constants $U_b, D_M$ and $\gamma$ are the front velocity for an ideal cylindrical front, the Markstein diffusivity and the thermal expansion coefficient respectively. $R_0(t)$ is the mean radius of the propagating flame:

$$R_0(t)=\frac{1}{2\pi}\int_0^{2\pi}R(\theta,t)d\theta\ . \tag{2}$$

The functional $I(R)$ is best represented in terms of its Fourier decomposition. Its Fourier component is $|k|R_k$ where $R_k$ is the Fourier component of $R$.

Numerical simulations of the type reported in ref. [11] are presented in Fig.1. The flame front $R(\theta,t)$ is shown at four equal time intervals. The two most prominent features of these simulations are the wrinkled multi-cusp appearance of the fronts and its acceleration as time progresses. One observes the phenomenon of tip splitting in which new cusps are added to the growing fronts between existing cusps. Both experiments and simulations indicate that for large times $R_0$ grows as a power in time

$$R_0(t)=(const+t)^\beta\ , \tag{3}$$

with $\beta>1$, (of the order of 1.5) and that the width of the interface $W$ grows with $R_0$ as

$$W(t)\sim R_0(t)^\chi\ , \tag{4}$$

with $\chi < 1$ (of the order of 2/3). The understanding of these two features and the derivation of the scaling relation between $\beta$ and $\chi$ are the main aims of this Letter.

Equation (2) can be written as a one-parameter equation by rescaling $R$ and $t$ according to $r \equiv RU_b/D_M$, $\tau \equiv tU_b^2/D_M$. Computing the derivative of Eq.(2) with respect to $\theta$ and substituting the dimensionless variables one obtains:

$$\frac{\partial u}{\partial \tau} = \frac{u}{r_0^2}\frac{\partial u}{\partial \theta} + \frac{1}{r_0^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\gamma}{2r_0} I\{u\} \ . \quad (5)$$

where $u \equiv \frac{\partial r}{\partial \theta}$. To complete this equation we need a second one for $r_0(t)$, which is obtained by averaging (2) over the angles and rescaling as above. The result is

$$\frac{dr_0}{d\tau} = \frac{1}{2r_0^2}\frac{1}{2\pi}\int_0^{2\pi} u^2 d\theta + 1 \ . \quad (6)$$

These two equations are the basis for further analysis

Following [14–19] we expand now the solutions $u(\theta, \tau)$ in poles whose position $z_j(\tau) \equiv x_j(\tau) + iy_j(\tau)$ in the complex plane is time dependent:

$$u(\theta,\tau) = \sum_{j=1}^{N} \cot\left[\frac{\theta - z_j(\tau)}{2}\right] + c.c. \quad (7)$$
$$= \sum_{j=1}^{N} \frac{2\sin[\theta - x_j(\tau)]}{\cosh[y_j(\tau)] - \cos[\theta - x_j(\tau)]} \ ,$$

$$r(\theta,\tau) = 2\sum_{j=1}^{N} \ln\Big[\cosh(y_j(\tau)) - \cos(\theta - x_j(\tau))\Big] + C(\tau) \ . \quad (8)$$

In (8) $C(\tau)$ is a function of time. The function (8) is a superposition of quasi-cusps (i.e. cusps that are rounded at the tip). The real part of the pole position (i.e. $x_j$) describes the angle coordinate of the maximum of the quasi-cusp, and the imaginary part of the pole position (i.e $y_j$) is related the height of the quasi-cusp. As $y_j$ decreases (increases) the height of the cusp increases (decreases). The physical motivation for this representation of the solutions should be evident from Fig.1.

The main advantage of this representation is that the propagation and wrinkling of the front can be described now via the dynamics of the poles and of $r_0(t)$. Substituting (7) in (5) we derive the following ordinary differential equations for the positions of the poles:

$$-r_0^2\frac{dz_j}{d\tau} = \sum_{k=1,k\neq j}^{2N} \cot\left(\frac{z_j - z_k}{2}\right) + i\frac{\gamma r_0}{2} sign[Im(z_j)] \ . \quad (9)$$

After substitution of (7) in (6) we get, using (9) the ordinary differential equation for $r_0$,

$$\frac{dr_0}{d\tau} = 2\sum_{k=1}^{N}\frac{dy_k}{d\tau} + 2\left(\frac{\gamma}{2}\frac{N}{r_0} - \frac{N^2}{r_0^2}\right) + 1 \ . \quad (10)$$

In the case of flame fronts propagating in channels of width $L$ ref. [15] presented a rather complete analysis of the available stationary solutions. Some aspects of this analysis are important also for our case of cylindrical geometry, and we therefore briefly summarize the main results of [15]. These are: (i) In noiseless conditions the total number of poles $N_T$ is conserved by the dynamics. This is also the case in the present problem. (ii) There is only one stable stationary solution which is geometrically represented by a giant cusp (or equivalently one finger) and analytically by $N(L)$ poles which are aligned on one line parallel to the imaginary axis. (iii) The reason for this behaviour is the existence of an attraction between the poles along the real line, and the resulting dynamics merges all the $x$ positions. The $y$ positions are distinct, and the poles are sitting above each others in positions $y_{j-1} < y_j < y_{j+1}$ with the maximal $y_{N(L)}$. (iv) If one adds an additional pole to such a solution, this pole (or another) will be pushed to infinity along the imaginary axis. If the system has less than $N(L)$ poles it is unstable to the addition of poles, and any noise will drive the system towards this unique state. The number $N(L)$ is

$$N(L) = \left[\frac{1}{2}\left(\frac{L}{2\pi\nu} + 1\right)\right] \ , \quad (11)$$

where $\Big[\dots\Big]$ is the integer part, and in the (different) parametrization of ref. [15] $\nu$ is the coefficient of the viscous term. (v) The height of the cusp is proportional to $L$. we will refer to the solution with these properties as the Thual-Frisch-Henon (TFH)-cusp solution.

In our problem the outward growth introduces important modifications to the channel results. The number of poles in a stable configuration is proportional here to the radius $r_0$ instead of $L$, but the former grows in time. The system becomes therefore unstable to the addition of new poles. If there is noise in the system that can generate new poles, they will not be pushed toward infinite $y$. It is important to stress that any infinitesimal noise (either numerical or experimental) is sufficient to generate new poles. These new poles do not necessarily merge their $x$-positions with existing cusps. Even though there is attraction along the real axis as in the channel case, there is a stretching of the distance between the poles due to the radial growth. This may counterbalance the attraction. Our first new idea is that these two opposing tendencies define a typical scale denoted as $\mathcal{L}$. if we have a cusp that is made from the $x$-merging of $N_c$ poles on the line $x = x_c$ and we want to know whether a $x$-nearby pole with real coordinate $x_1$ will merge with this large cusp, the answer depends on the distance $D = r_0|x_c - x_1|$. There is a length $\mathcal{L}(N_c, r_0)$ such that if $D > \mathcal{L}(N_c, r_0)$ then the single cusp will never merge with the larger cusp. In the

opposite limit the single cusp will move towards the large cusp until their $x$- position merges and the large cusp will have $N_c + 1$ poles.

This finding stems directly from the equations of motion of the $N_c$ $x$-merged poles and the single pole at $x_1$. First note that from Eq.3 (which is not explained yet) it follows that asymptotically $r_0(\tau) = (a+\tau)^\beta$ where $r_0(0) = a^\beta$. Next start from 9 and write equations for the angular distance $x = x_1 - x_c$. It follows that for any configuration $y_j$ along the imaginary axis

$$\frac{dx}{d\tau} \leq -\frac{2N_c \sin x[1-\cos x]^{-1}}{(a+\tau)^{2\beta}} = -\frac{2N_c \cot(\frac{x}{2})}{(a+\tau)^{2\beta}} \ . \quad (12)$$

For small x we get

$$\frac{dx}{d\tau} \leq -\frac{4N_c}{x(a+\tau)^{2\beta}} \ . \quad (13)$$

The solution of this equation is

$$x(0)^2 - x(\tau)^2 \geq \frac{8N_c}{2\beta - 1}(a^{1-2\beta} - (a+\tau)^{1-2\beta}) \ . \quad (14)$$

To find $\mathcal{L}$ we set $x(\infty) > 0$ from which we find that the angular distance will remain finite as long as

$$x(0)^2 > \frac{8N_c}{2\beta - 1} a^{1-2\beta} \ . \quad (15)$$

Since $r_0 \sim a^\beta$ we find the threshold angle $x^*$

$$x^* \sim \sqrt{N_c} r_0^{\frac{(1-2\beta)}{2\beta}} \ , \quad (16)$$

above which there is no merging between the giant cusp and the isolated pole. To find the actual distance $\mathcal{L}(N_c, r_0)$ we multiply the angular distance by $r_0$ and find

$$\mathcal{L}(N_c, r_0) \equiv r_0 x^* \sim \sqrt{N_c} r_0^{\frac{1}{2\beta}} \ . \quad (17)$$

To understand the geometric meaning of this result we recall the features of the TFH cusp solution. Having a typical length $L$ the number of poles in the cusp is linear in $L$. Similarly, if we have in this problem two cusps a distance $2\mathcal{L}$ apart, the number $N_c$ in each of them will be of the order of $\mathcal{L}$. From (17) it follows that

$$\mathcal{L} \sim r_0^{\frac{1}{\beta}} \ . \quad (18)$$

For $\beta > 1$ the circumference grows faster than $\mathcal{L}$, and therefore at some points in time poles that appear between two large cusps would not be attracted toward either, and new cusps will appear. We will show later that the most unstable positions to the appearance of new cusps are precisely the midpoints between existing cusps. This is the mechanism for the addition of cusps in analogy with tip splitting in Laplacian growth.

We can now estimate the width of the flame front as the height of the largest cusps. Since this height is proportional to $\mathcal{L}$ (cf. property (v) of the TFH solution), Eq.(18) and Eq.(4) lead to the scaling relation

$$\chi = 1/\beta \ . \quad (19)$$

This scaling law is expected to hold all the way to $\beta = 1$ for which the flame front does not accelerate and the size of the cusps becomes proportional to $r_0$.

Next we shed light on phenomenon of tip splitting that here is seen as the addition of new cusps roughly in between existing ones. We mentioned the instability toward the addition of new poles. We argue now that the tip between the cusps is most sensitive to pole creation. This can be shown in both channel and radial geometry. For example consider a TFH-giant cusp solution in which all the poles are aligned (without loss of generality) on the $x = 0$ line. Add a new pole in the complex position $(x_a, y_a)$ to the existing $N(L)$ poles, and study its fate. It can be shown that in the limit $y_a \to \infty$ (which is the limit of a vanishing perturbation of the solution) the equation of motion is

$$\frac{dy_a}{d\tau} = \frac{2\pi\nu}{L}(2N(L)+1) - 1 \qquad y_a \to \infty \ . \quad (20)$$

Since $N(L)$ satisfies (11) this equation can be rewritten as

$$\frac{dy_a}{d\tau} = \frac{4\pi\nu}{L}(1-\alpha) \qquad y_a \to \infty \ , \quad (21)$$

where $\alpha = (L/(2\pi\nu)+1)/2 - N(L)$. Obviously $\alpha \leq 1$ and it is precisely 1 only when $L$ is $L = (2n+1)2\pi\nu$. Next it can be shown that for $y_a$ much larger than $y_{N(L)}$ but not infinite the following is true:

$$\frac{dy_a}{d\tau} > \lim_{y_a \to \infty} \frac{dy_a}{d\tau} \quad x_a = 0 \quad (22)$$

$$\frac{dy_a}{d\tau} < \lim_{y_a \to \infty} \frac{dy_a}{d\tau} \quad x_a = \pi \quad (23)$$

We learn from these results that there exist values of $L$ for which a pole that is added at infinity will have marginal attraction ($dy_a/dt = 0$). Similar understanding can be obtained from a standard stability analysis without using pole decomposition. Perturbing a TFH-cusp solution we find linear equations whose eigenvalues $\lambda_i$ can be obtained by standard numerical techniques. Fig.2 presents $Re(\lambda_i)$ as a function of systems size, and shows very clearly that (i) all $Re(\lambda_i)$ are non-positive. (ii) at the isolated values of $L$ for which $L = (2n+1)2\pi\nu$ $Re(\lambda_1)$ and $Re(\lambda_2)$ become zero (note that due to the logarithmic scale the zero is not evident) (iii) There exists a general tendency of all $Re(\lambda_i)$ to approach zero from below as $L$ increases. This indicates a growing sensitivity to noise when the system size increases. (iv) There exists a Goldsone mode $\lambda_0 = 0$ due to translational invariance.

The upshot of this discussion is that finite perturbations (i.e. poles at finite $y_a$) will grow if the $x$ position of the pole is sufficiently near the tip. The position $x = \pi$ (the tip of the finger) is the most unstable one. In the channel geometry this means that noise results in the appearance of new cusps at the tip of the fingers, but due to the attraction to the giant cusp they move toward $x = 0$ and disappear in the giant cusp. In fact, one sees in numerical simulations a train of small cusps that move toward the giant cusp. Analysis shows that at the same time the furthest pole at $y_{N(L)}$ is pushed towards infinity. Also in cylindrical geometry the most sensitive position to the appearance of new cusps is right between two existing cusps independently if the system is marginal (the total number of poles fits the radius) or unstable (total number of poles is too small at a given radius). Whether or not the addition of a new pole results in tip splitting depends on their $x$ position. When the distance from existing cusps is larger than $\mathcal{L}$ the new poles that are generated by noise will remain near the tip between the two cusps and will cause tip splitting.

Lastly, we note that without the noisy generation of new poles acceleration is impossible. This is seen directly from Eq.(10) in which all the terms on the RHS but unity go to zero with $r_0 \to \infty$, and the velocity saturates. We need the noisy appearance of new poles to achieve acceleration. The precise connection between the noise amplitude, system size and acceleration that leads to the computation of the exponent $\beta$ is beyond the scope of this letter and will be discussed in a forthcoming publication.

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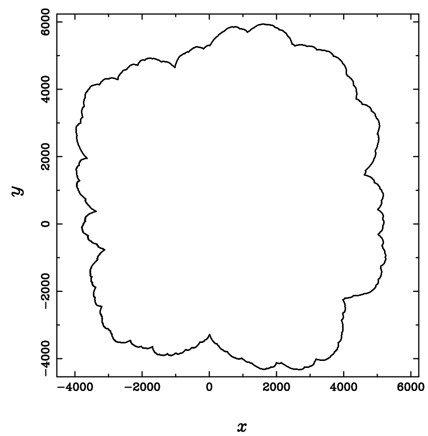


FIG. 1. Simulations of the outward propagating flame front. Note that deep cusps do not disappear and that new deep cusps appear when the rounded tips split.

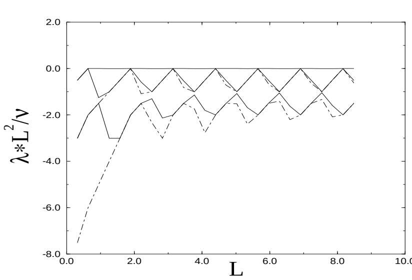


FIG. 2. Successive eigenvalues of the stability matrix of the TFH-giant cusp solution as a function of the system size $L$. The leading eigenvalue touches zero periodically in $L$. All the eigenvalues tend to zero when $L \to \infty$ as $L^{-2}$.